

Worcester County Mathematics League

Varsity Meet 3 - January 28, 2026

COACHES' COPY ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 3 - January 28, 2026
Answer Key



Round 1 Similarity and Pythagorean Theorem

1. 14
2. $\frac{5}{2}$ or $2\frac{1}{2}$ or 2.5
3. $21\sqrt{2}$

Team Round

1. 18
2. 5
3. 4

Round 2 - Algebra I

1. 22
2. 44
3. 24.5 or $24\frac{1}{2}$ or $\frac{49}{2}$

4. $(-2, -2)$
5. (99, 80) exact order
6. 55
7. $\left(\frac{1}{2}, -\frac{3}{2}\right)$ or $\left(\frac{1}{2}, -1\frac{1}{2}\right)$ or $(0.5, -1.5)$

Round 3 - Functions

1. 2
2. -26
3. $(-6, -4)$ exact order

8. 48
9. $\sqrt{61}$

Round 4 - Combinatorics

1. 20
2. 54
3. 166

Round 5 - Analytic Geometry

1. 10
2. $(-8, 0)$ and $\left(\frac{8}{3}, 0\right)$ each pair ordered exactly, need both pairs, either listed first. $2\frac{2}{3}$ and $2.\overline{6}$ are acceptable forms of $\frac{8}{3}$.
3. $\left(-\frac{1}{60}, \frac{5}{12}, \frac{3}{5}\right)$ exact order

Worcester County Mathematics League
Varsity Meet 3 - January 28, 2026
Round 1 - Similarity and Pythagorean Theorem

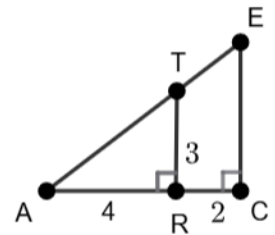


All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Given $\triangle MOT \sim \triangle FEN$, $MO = 6$, $OT = 4$, $TM = 3$, and $NF = 7$: find the length of the longest side of $\triangle FEN$.

2. Given $\triangle ACE$ in the figure shown at right, where $RT = 3$, $AR = 4$, and $RC = 2$: find TE .



3. Given quadrilateral $ABCD$ where $AB = 2\sqrt{3}$, $BC = 3\sqrt{6}$, $CD = \sqrt{66}$, $DA = 2\sqrt{6}$, $\overline{AC} \perp \overline{BD}$, $\overline{AC}$ intersects $\overline{BD}$ at E , and DE is an integer: find the area of $ABCD$. Express your answer in simplest radical form.

ANSWERS

(1 pt) 1. _____

(2 pts) 2. $TE =$ _____

(3 pts) 3. _____

Worcester County Mathematics League
Varsity Meet 3 - January 28, 2026
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Solve for x :

$$7 + 3\sqrt{3x - 2} = 31$$

2. In 1930 a man wrote to a newspaper about a friend who had died on Dec. 31 in an unspecified year. He first met his friend on Dec. 31, 1900. Her age (a whole number of years) at the time of her death was exactly one twenty-ninth of her birth year. What was the friend's age, in years, when they first met?
3. Farmers Brown and Green want to enclose a rectangular area using $42ft$ of fencing where one side of the enclosure is a side of a barn that is $30ft$ in length. Brown wants to enclose a square, but Green thinks that a larger area could be enclosed. What is the difference, measured in ft^2 , between the largest possible rectangular area than can be enclosed and a square area that uses all $42ft$ of fencing?

ANSWERS

(1 pt) 1. $x =$ _____

(2 pts) 2. _____ years

(3 pts) 3. _____ ft^2

Southbridge, Holy Name, Quaboag/QSC

Worcester County Mathematics League
Varsity Meet 3 - January 28, 2026
Round 3 - Functions



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Let $f(x) = \sqrt{\frac{x+2}{x-4}}$. Find $f(6)$.

2. If $f(x) = x^2 + bx + c$ and $f(2026 - x) = f(x - 2000)$, find b .

3. Let $f(x) = \frac{1}{x}$ and let $g(x) = f\left(\frac{x}{18}\right)$. That is, the graph of $y = g(x)$ on the x, y plane is the graph of $y = f(x)$ vertically stretched by a factor of 18. Let $h(x)$ be the function whose graph is a translation of the graph of $y = g(x)$ by 3 units to the right and 2 units up. If $y = h(x)$ intersects the x -axis at $(x_0, 0)$ and the y -axis at $(0, y_0)$, find the ordered pair (x_0, y_0) .

ANSWERS

(1 pt) _____

(2 pts) 2. $b =$ _____

(3 pts) 3. $(x_0, y_0) =$ _____

Bartlett, Bromfield, QSC

Worcester County Mathematics League
Varsity Meet 3 - January 28, 2026
Round 4 - Combinatorics



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Six points lie on the circumference of a circle. How many distinct triangles can be drawn having three of these points as vertices? Note: three points that do not lie on a line determine a triangle.
2. How many ways are there to select four people from four couples where the four chosen people include at least one couple?
3. How many distinct three letter "words" can be formed from the letters in the word "*bookkeeper*"? For example, three such "words" are "kek", "kke", and "ore", but not "bob"; "bob" cannot be formed from the letters of *bookkeeper* because there is only one "b" in *bookkeeper*.

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Westborough, Westborough, Shrewsbury/QSC

Worcester County Mathematics League
Varsity Meet 3 - January 28, 2026
Round 5 - Analytic Geometry



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find the area of the triangle bounded by the lines $x = 0$, $y = 0$, and $5x + 4y = 20$.
2. Joan draws the locus of points P such that the ratio $(d_0(P) : d_8(P)) = (1 : 2)$, where $d_0(P)$ is the distance of P to $(0, 0)$ and $d_8(P)$ is the distance of P to $(8, 0)$. Find the coordinates of all points (x, y) where the locus intersects with the x -axis.
3. The parabola $y = x^2$ passes through the three points $(1, 1)$, $(2, 4)$, and $(3, 9)$. If the parabola $x = ay^2 + by + c$ also passes through these points, find (a, b, c) , where a , b and c are expressed as common fractions in reduced form (a ratio of integers $\frac{m}{n}$).

ANSWERS

(1 pt) 1. _____

(2 pts) 2. $\left\{ \rule{1.5cm}{0.4pt} \right\}$

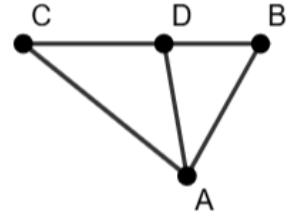
(3 pts) 3. $(a, b, c) = \left(\rule{1.5cm}{0.4pt} \right)$

Worcester County Mathematics League
Varsity Meet 3 - January 28, 2026
Team Round



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED



1. In $\triangle ABC$, shown in the figure to the right, $\overline{AD}$ bisects $\angle CAB$ and D lies on $\overline{BC}$ between C and B . If $AC = 36$, $AB = 27$, and $BC = 42$, find BD .

2. Penelope has dimes and quarters in her pocket worth a total of \$1.15. What is the difference between the largest number and the smallest number of dimes that she might have in her pocket?
3. Let $f(x)$ and $g(x)$ be functions with domain $D = \{1, 2, 3, 4, 5\}$. They are defined by the following table:

x	1	2	3	4	5
$f(x)$	4	5	2	3	1
$g(x)$	3	1	4	5	2

Find $f(g^{-1}(f^{-1}(2)))$.

4. Point S has coordinates $(x, y) = (-2, 4)$. When S is rotated 90° counterclockwise around the point $(1, 1)$ its coordinates are (x_{90}, y_{90}) . Find (x_{90}, y_{90}) .
5. Square $ABCD$ is located in the first quadrant of the x - y plane. A lies on the y -axis, B lies on the x -axis, and the coordinates of D are $(19, 99)$. Find (x_C, y_C) , the coordinates of C .
6. At 8am Andrew started driving his Audi due North from his home in Beantown on Route 314. At 12 noon Cindy started driving her Cadillac from Andrew's home, following Andrew on Route 314. At 5pm Cindy had passed Andrew and was 50 miles ahead of him. If Andrew averaged 25 miles per hour (mph) for his drive, what was Cindy's average speed, measured in mph?
7. If $f(x) = 2x + 3$, then $f^{-1}(x) = ax + b$, where $f^{-1}(x)$ is the inverse of $f(x)$. Find the ordered pair (a, b) where a and b are expressed as fractions.
8. How many ways can three boys and two girls sit in a row of five chairs if the two girls always sit next to each other?
9. What is the radius of the circle that passes through the points $(0, 0)$, $(1, 1)$, and $(11, 1)$? Give your answer in simplest radical form.

Worcester County Mathematics League
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Team Round Answer Sheet



ANSWERS

1. $BD =$ _____

2. _____ dimes

3. _____

4. $(x_{90}, y_{90}) = ($ _____ $)$

5. $(x_C, y_C) = ($ _____ $)$

6. _____ mph

7. $(a, b) = ($ _____ $)$

8. _____

9. _____

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Answer Key



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Round 1 - Similarity and Pythagorean Theorem

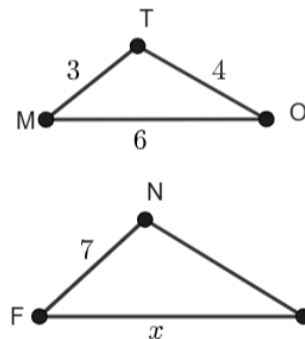
1. Given $\triangle MOT \sim \triangle FEN$, $MO = 6$, $OT = 4$, $TM = 3$, and $NF = 7$: find the length of the longest side of $\triangle FEN$.

Solution: First, draw the two triangles in the same orientation to visualize the proportionality of the segment lengths, as shown to the right. Next, write the proportions derived from corresponding segments matched based on their position in the given similarity statement ($\triangle MOT \sim \triangle FEN$):

$$\frac{FE}{MO} = \frac{EN}{OT} = \frac{NF}{TM} = s$$

and $s = \frac{FE}{MO} = \frac{7}{3}$ is the scale factor from $\triangle MOT$ to $\triangle FEN$.

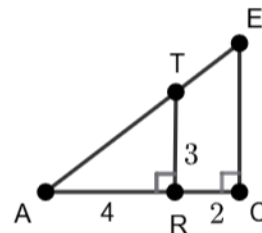
Note that the longest side in $\triangle MOT$ is $MO = 6$ and that the longest side in $\triangle FEN$ is the side that corresponds to $\overline{MO}$, namely $\overline{FE}$. Then $FE = s \cdot MO = \frac{7}{3} \cdot 6 = 7 \cdot \frac{6}{3} = 7 \cdot 2 = \boxed{14}$.



2. Given $\triangle ACE$ in the figure shown at right, where $RT = 3$, $AR = 4$, and $RC = 2$: find TE .

Solution: Recall the Triangle Proportionality Theorem: a line segment drawn parallel to one side of the triangle and intersecting the other two sides will divide the two sides in proportion. Therefore $\frac{TE}{AT} = \frac{RC}{AR}$, or

$$TE = AT \cdot \frac{RC}{AR}.$$



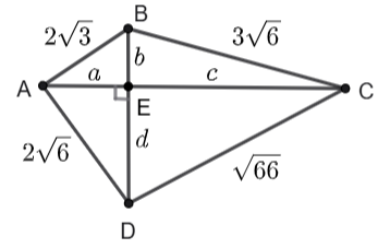
Solution: (continued) Observe that $\triangle ART$ is a 3-4-5 right triangle with hypotenuse AT , so

$AT = 5$. Substitute the values for AT , RC and AR into the above equation: $TE = 5 \cdot \frac{2}{4} = 5 \cdot \frac{1}{2} = \boxed{\frac{5}{2}}$.

Alternative solution: First, note that $\triangle ACE \sim \triangle ART$ by AA similarity: $\angle ART \cong \angle ACE$ because they are both right angles, and $\angle A$ is shared between the two triangles. Therefore $\frac{AE}{AT} = \frac{AC}{AR}$ because corresponding sides of similar triangles are in proportion. Next, apply the Pythagorean Theorem to find AT : $AT^2 = AR^2 + RT^2 = 4^2 + 3^2 = 16 + 9 = 25$, and $AT = \sqrt{25} = 5$. Note that $AC = AR + RC = 4 + 2 = 6$ and substitute the values for AR , AC , and AT into the proportion to find AE : $\frac{AE}{5} = \frac{6}{4} = \frac{3}{2}$ and $AE = 5 \cdot \frac{3}{2} = \frac{15}{2}$. Finally, $AE = AT + TE$, so $TE = AE - AT = \frac{15}{2} - 5 = \frac{15-10}{2} = \frac{5}{2}$, as above.

3. Given quadrilateral $ABCD$ where $AB = 2\sqrt{3}$, $BC = 3\sqrt{6}$, $CD = \sqrt{66}$, $DA = 2\sqrt{6}$, $\overline{AC} \perp \overline{BD}$, $\overline{AC}$ intersects $\overline{BD}$ at E , and DE is an integer: find the area of $ABCD$. Express your answer in simplest radical form.

Solution: First sketch $ABCD$ such that diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. Label the drawing with given information, as shown on the right. Also label segment lengths $AE = a$, $BE = b$, $CE = c$ and $DE = d$. Note that $\triangle AEB$, $\triangle BEC$, $\triangle CED$, and $\triangle DEA$ are right triangles. Apply the Pythagorean Theorem to each of these triangles:



$$a^2 + b^2 = (2\sqrt{3})^2 = 4 \cdot 3 = 12$$

$$b^2 + c^2 = (3\sqrt{6})^2 = 9 \cdot 6 = 54$$

$$c^2 + d^2 = (\sqrt{66})^2 = 66$$

$$d^2 + a^2 = (2\sqrt{6})^2 = 4 \cdot 6 = 24.$$

Note that $a^2 > 0$. Therefore $d^2 < 24$ because $a^2 + d^2 = 24$. Also, $d^2 < 5^2$ because $24 < 25 = 5^2$.

Next, subtract the first equation from the last: $d^2 + a^2 - a^2 - b^2 = d^2 - b^2 = 24 - 12 = 12$. Add b^2 to both sides of this equation and $d^2 = b^2 + 12$. Note that $b^2 > 0$, so, $d^2 > 12 > 3^2$. Put the two inequalities together: $3^2 < d^2 < 5^2$. The only positive integer solution to this inequality is $d = 4$.

Proceed by solving for a , b , and c using three of the four equations. From the fourth equation: $a^2 = 24 - d^2 = 24 - 16 = 8$ because $d^2 = 4^2 = 16$. Then $a = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$. From the third equation, $c^2 = 66 - d^2 = 66 - 16 = 50$, and $c = \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$. From the second equation, $b^2 = 54 - c^2 = 54 - 50 = 4$, and $b = \sqrt{4} = 2$.

Denote the area of a polygon as $[.]$, where “.” is the name of the polygon. Then $[ABCD] = [\triangle ABC] + [\triangle ADC]$ (the quadrilateral area is equal to the sum of the areas of the two triangles created by the diagonal $\overline{AC}$). Recall that the area of a triangle is equal to $\frac{1}{2}bh$, one half the base length times its height. In this case, the base length for both triangles is $\overline{AC} = a + c$, and the heights of the two triangles are b and d . Then

$$[ABCD] = [\triangle ABC] + [\triangle ADC] = \frac{1}{2}(a + c)b + \frac{1}{2}(a + c)d = \frac{1}{2}(a + c)(b + d).$$

Substitute the known values of a , b , c , d : $[ABCD] = \frac{1}{2}(2 + 4)(2\sqrt{2} + 5\sqrt{2}) = \frac{1}{2} \cdot 6 \cdot 7\sqrt{2} = 3 \cdot 7\sqrt{2} = \boxed{21\sqrt{2}}$.

Round 2 - Algebra I

1. Solve for x :

$$7 + 3\sqrt{3x - 2} = 31$$

Solution: Begin by subtracting 7 from both sides of the equation: $3\sqrt{3x - 2} = 31 - 7 = 24$. Divide both sides of this equation by 3, then square both sides of the resulting equation:

$$\sqrt{3x - 2} = \frac{24}{3} = 8.$$

$$3x - 2 = 8^2 = 64.$$

Add 2 to both sides: $3x = 66$. Finally, divide by 3: $x = \frac{66}{3} = \boxed{22}$.

2. In 1930 a man wrote to a newspaper about a friend who had died on Dec. 31 in an unspecified year. He first met his friend on Dec. 31, 1900. Her age (a whole number of years) at the time of her death was exactly one twenty-ninth of her birth year. What was the friend's age, in years, when they first met?

Solution: Let B be the birth year of the friend, D be the year that she died, a be her age at the time of death, and m be her age when the man met her. Note: because the dates of meeting and death are both Dec. 31, $B + m = 1900$ and $B + a = D$ regardless of the friend's birthday.

It was given that $a = \frac{B}{29}$. Thus, $B = 29a$, $D = B + a = 29a + a = 30a$, and the year of her death must be divisible by 30. The only number divisible by 30 between 1900 and 1930 is 1920. Therefore $D = 1920$, $a = \frac{1920}{30} = 64$, $B = 1920 - 64 = 1856$, and $m = 1900 - 1856 = \boxed{44}$.

3. Farmers Brown and Green want to enclose a rectangular area using 42 ft of fencing where one side of the enclosure is a side of a barn that is 30 ft in length. Brown wants to enclose a square, but Green thinks that a larger area could be enclosed. What is the difference, measured in ft^2 , between the largest possible rectangular area than can be enclosed and a square area that uses all 42 ft of fencing?

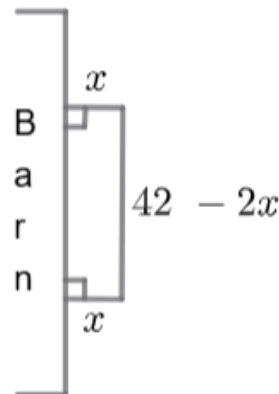
Solution: Draw and label the barn and enclosed area, as shown at right.

Let x be the length of the two sides that are perpendicular to the barn. Then the length of the side parallel to the barn, in ft , is equal to $42 - 2x$ when all the fencing is used. Note that the area of the rectangle, A , is a function of x , and $A = f(x) = x(42 - 2x) = 2x(21 - x)$.

Observe that the graph of $f(x)$ is a parabola opening down with x -intercepts at $(x, f(x)) = (0, 0)$ and $(21, 0)$. Recall that the parabola reaches its maximum value at its vertex, which lies on the axis of symmetry of the parabola. Note that the axis of symmetry lies halfway between the two x -intercepts, or at $x = \frac{21+0}{2} = \frac{21}{2}$. Substitute this value for x to find the maximum area of a rectangular enclosure: $A = f(x) = \frac{21}{2}(42 - 2 \cdot \frac{21}{2}) = \frac{21}{2}(42 - 21) = \frac{21}{2} \cdot 21 = \frac{441}{2}$.

Next, observe that if the enclosed area is square, then $x = 42 - 2x$ because all sides are equal. Solve for x : $x + 2x = 3x = 42$, so $x = \frac{42}{3} = 14$. The area of the square enclosure is therefore $14^2 = 196$.

The difference between the largest area and the square area is $\frac{441}{2} - 196 = \frac{441-392}{2} = \boxed{\frac{49}{2}} \text{ ft}^2$.



Round 3 - Functions

1. **(Functions)** Let $f(x) = \sqrt{\frac{x+2}{x-4}}$. Find $f(6)$.

Solution: Replace each instance of x with 6 in the definition of $f(x)$ and simplify the expression:

$$f(6) = \sqrt{\frac{6+2}{6-4}} = \sqrt{\frac{8}{2}} = \sqrt{4} = \boxed{2}$$

2. If $f(x) = x^2 + bx + c$ and $f(2026 - x) = f(x - 2000)$, find b .

Solution: Note that the graph of $f(x)$ is a parabola opening upwards. Recall that the graph of such a parabola has line of symmetry at $x = k$ such that $f(k - x) = f(x + k)$ for any real number x . Substitute $x' = x + 2013$ in the given equality $f(2026 - x) = f(x - 2000)$:

$$f(2026 - (x' + 2013)) = f((x' + 2013) - 2000)$$

$$f(2026 - x' - 2013) = f(x' + 2013 - 2000)$$

$$f(13 - x') = f(x' + 13)$$

and observe that $f(x')$ has an axis of symmetry at $x' = k = 13$. Also, observe that $f(x')$ is the same function as $f(x)$; the only change was the name of the dependent variable. Therefore $f(13 - x) = f(x + 13)$ and $f(x)$ has an axis of symmetry at $x = 13$.

Note: if the graph of $f(x) = x^2 + bx + c$ is a translation of $g(x) = x^2$ by h units to the right and k units up, then

$$f(x) = g(x - h) + k = (x - h)^2 + k = x^2 - 2hx + h^2 + k$$

Recall that the vertex of $g(x) = x^2$ is $(0, 0)$, and therefore that the vertex of $f(x)$ is equal to (h, k) (due to the translation h right and k up). Also recall that the vertex lies on the axis of symmetry, so $h = 13$. Set $f(x) = x^2 + bx + c = x^2 - 2hx + h^2 + k$ and $b = -2h = -2(13) = \boxed{-26}$.

3. Let $f(x) = \frac{1}{x}$ and let $g(x) = f\left(\frac{x}{18}\right)$. That is, the graph of $y = g(x)$ on the x, y plane is the graph of $y = f(x)$ vertically stretched by a factor of 18. Let $h(x)$ be the function whose graph is a translation of the graph of $y = g(x)$ by 3 units to the right and 2 units up. If $y = h(x)$ intersects the x -axis at $(x_0, 0)$ and the y -axis at $(0, y_0)$, find the ordered pair (x_0, y_0) .

Solution: Recall that if the graph of $h(x)$ is a translation of $g(x)$ (horizontally) to the right by m units and (vertically) up n units, then $h(x) = g(x - m) + n$. Then $g(x - 3) = f\left(\frac{x-3}{18}\right)$ and

$$\begin{aligned} h(x) &= g(x - 3) + 2 \\ &= f\left(\frac{x - 3}{18}\right) + 2 \\ &= \frac{18}{x - 3} + 2. \end{aligned}$$

Substitute $h(x) = 0$ and $x = x_0$ into the last equation to find x_0 :

$$\begin{aligned} 0 &= \frac{18}{x_0 - 3} + 2 \\ -2 &= \frac{18}{x_0 - 3} \\ x_0 - 3 &= \frac{18}{-2} = -9 \\ x_0 &= -9 + 3 = -6 \end{aligned}$$

Next, substitute $h(x) = y_0$ and $x = 0$ into the same expression to find y_0 :

$$y_0 = \frac{18}{0 - 3} + 2 = \frac{18}{-3} + 2 = -6 + 2 = -4$$

Thus $(x_0, y_0) = \boxed{(-6, -4)}$.

Round 4 - Combinatorics

1. Six points lie on the circumference of a circle. How many distinct triangles can be drawn having three of these points as vertices? Note: three points that do not lie on a line determine a triangle.

Solution: Note that three points of a circle cannot lie on one line. Therefore the number of triangles that can be drawn equals the number of ways to choose three of the six points, *without regard to the order in which they are chosen*. If the order mattered, then there would be 6 choices for the first point, 5 choices for the second point, and 4 choices for the third point, giving a total of ${}_6P_3 = 6 \cdot 5 \cdot 4$ permutations of three points chosen from six. This number overcounts the number of triangles because each choice of three points can be ordered ${}_3P_3 = 3 \cdot 2 \cdot 1$ ways. Therefore the number of triangles is equal to $\frac{{}_6P_3}{{}_3P_3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 5 \cdot 4 \cdot \frac{6}{6} = \boxed{20}$.

Alternative solution: The number of triangles is the number of combinations of three points chosen from the original set of six points, or ${}_6C_3$, where ${}_nC_k = \frac{n!}{(n-k)!k!}$ and $n! = n(n-1) \cdot \dots \cdot (2)(1)$. Then ${}_6C_3 = \frac{6!}{(6-3)!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$ as in the above solution. (! is the factorial symbol.)

2. How many ways are there to select four people from four couples where the four chosen people include at least one couple?

Solution: First note that there are 8 individual people in 4 couples.

This problem is most easily solved using the method of complementary counting. Let N equal the number of ways to select 4 out of 8 people. Let M equal the number of ways to select four people from four couples where no two of the four people are a couple. Then the number of ways to choose four people, including at least one couple, from 4 couples is $N - M$.

N is equal the number of ways to choose 4 of 8, or

$$N = {}_8C_4 = \frac{8!}{(8-4)!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!(4 \cdot 3 \cdot 2 \cdot 1)} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 2 \cdot 3} = \frac{8 \cdot 3 \cdot 2 \cdot 7 \cdot 5}{8 \cdot 3} = 2 \cdot 7 \cdot 5 = 70$$

M is given by the number of ways to choose one person from each of the four couples, or $M = {}_2C_1 \cdot {}_2C_1 \cdot {}_2C_1 \cdot {}_2C_1 = ({}_2C_1)^4$. Note that ${}_2C_1$, the number of ways to choose 1 of 2 people, is equal to 2. Then $M = 2^4 = 16$ and $N - M = 70 - 16 = \boxed{54}$.

3. How many distinct three letter “words” can be formed from the letters in the word *bookkeeper*? For example, three such “words” are “kek”, “kke”, and “ore”, but not “bob”; “bob” cannot be formed from the letters of *bookkeeper* because there is only one “b” in *bookkeeper*.

Solution: First count the letters. There are 10 letters in *bookkeeper*. Of those 10 letters, there are two o’s, two k’s, three e’s, one b, one p, and one r. Count the number of words by dividing them into three cases.

Case 1: triple letter. There is one triple letter word, namely “eee”.

Case 2: a double (not necessarily consecutive) letter. There are three choices for the double letter: o, e, or k. For each of these choices, there are five choices for the third letter. For each of these choices of double and single letters, there are three ways to order the letters. For example, o, o, and r can be formed into “oor”, “oro”, and “roo”. Thus, there are $3 \cdot 5 \cdot 3 = 45$ possible three letter words with a double letter.

Case 3: three different letters. In this case, there are “6 choose 3”, or ${}_6C_3$ words formed from three different letters. Calculate ${}_6C_3 = \frac{6!}{(6-3)!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 5 \cdot 4 \cdot \frac{6}{6} = 20$. For each of the 20 combinations of three distinct letters, there are $3 \cdot 2 \cdot 1 = 6$ ways to order the letters. In total there are $20 \cdot 6 = 120$ words with three distinct letters.

Add up the totals from the three cases: $1 + 45 + 120 = \boxed{166}$.

Round 5 - Analytic Geometry

1. Find the area of the triangle bounded by the lines $x = 0$, $y = 0$, and $5x + 4y = 20$.

Solution: Observe that two of the sides of the triangle lie on the x -axis ($y = 0$) and the y -axis ($x = 0$). These sides lie at a right angle because the x -axis and the y -axis are perpendicular. Therefore the triangle is a right triangle, the right angle's vertex is the origin $((x, y) = (0, 0))$ and the line $5x + 4y = 20$ contains the hypotenuse. Also, the side lengths are determined by the x - and y -intercepts of the line $5x + 4y = 20$.

Find the x - and y -intercepts by converting the equation to the intercept form: $\frac{x}{a} + \frac{y}{b} = 1$, where $(a, 0)$ is the x -intercept and $(0, b)$ is the y -intercept. Divide both sides of $5x + 4y = 20$ by 20:

$$\frac{5x}{20} + \frac{4y}{20} = \frac{x}{4} + \frac{y}{5} = \frac{20}{20} = 1$$

and the intercepts are $(4, 0)$ and $(0, 5)$. The triangle therefore has side lengths 4 and 5. Recall that the area of a right triangle is one half the product of its side lengths: $\frac{1}{2} \cdot 4 \cdot 5 = 2 \cdot 5 = \boxed{10}$.

2. Joan draws the locus of points P such that the ratio $d_0(P) : d_8(P) = 1 : 2$, where $d_0(P)$ is the distance of P to $(0, 0)$ and $d_8(P)$ is the distance of P to $(8, 0)$. Find the coordinates of all points (x, y) where the locus intersects with the x -axis.

Solution: First, note that any point p in the intersection of the locus with the x -axis will lie on the x -axis, in line with $(0, 0)$ and $(8, 0)$. Its coordinates will be of the form $(x_k, 0)$. Find all points by testing each of three cases.

Case 1: $x_k < 0$. Then $d_0(P) = -x_k$, the (positive) distance from P to the origin. Also, $d_8(P) = -x_k + 8$: the distance from P to $(0, 8)$ is the sum of $d_0(P)$ and the distance of the origin to $(8, 0)$. Then $(d_0(P) : d_8(P)) = -x_k : (-x_k + 8) = 1 : 2$, or $\frac{-x_k}{-x_k + 8} = \frac{1}{2}$. Multiply both sides by $2(-x_k + 8)$:

$$\begin{aligned} -2x_k &= -x_k + 8 \\ -8 &= 2x_k - x_k = x_k \end{aligned}$$

and $x_k = -8$. The coordinates of P are $(-8, 0)$.

Case 2: $0 < x_k < 8$. Then $d_0(P) = x_k$ and $d_8(P) = 8 - x_k$. Express the proportion in fraction form: $\frac{d_0(P)}{d_8(P)} = \frac{x_k}{8 - x_k} = \frac{1}{2}$. Multiply both sides of the equation by the product of the denominators: $2x_k = 8 - x_k$. Add x_k to both sides: $3x_k = 8$, and $x_k = \frac{8}{3}$. The coordinates of P are $(\frac{8}{3}, 0)$.

Case 3: $8 < x_k$. In this case, P is closer to $(8, 0)$ than to $(0, 0)$ ($d_0(P) > d_8(P)$), the required ratio cannot be less than one, and there are no solutions for x_k .

Put the results together. Two points from the locus lie on the x -axis: $(-8, 0)$ and $(\frac{8}{3}, 0)$.

3. The parabola $y = x^2$ passes through the three points $(1, 1)$, $(2, 4)$, and $(3, 9)$. If the parabola $x = ay^2 + by + c$ also passes through these points, find (a, b, c) , where a , b and c are expressed as common fractions in reduced form (a ratio of integers $\frac{m}{n}$).

Solution: Set up a system of three equations in the three unknowns a, b, c by successively substituting the coordinates of each of the points in the equation $x = ay^2 + by + c$:

$$1 = a(1)^2 + b(1) + c = a + b + c$$

$$2 = a(4)^2 + b(4) + c = 16a + 4b + c$$

$$3 = a(9)^2 + b(9) + c = 81a + 9b + c$$

Solve this system by elimination. Subtract the first equation from the second and third equations:

$$1 = 15a + 3b$$

$$2 = 80a + 8b$$

and c is eliminated. Multiply the first equation by 8 and the second equation by 3 to eliminate b :

$$8 = 120a + 24b$$

$$6 = 240a + 24b.$$

Subtract the bottom equation from the top equation: $8 - 6 = 2 = (120 - 240)a + 24b - 24b = -120a$. Thus, $a = \frac{2}{-120} = -\frac{1}{60}$.

Proceed by substitution. Substitute $a = -\frac{1}{60}$ into the first equation of the group of two equations:

$$1 = 15 \left(-\frac{1}{60} \right) + 3b = -\frac{1}{4} + 3b$$

Then $3b - \frac{1}{4} = 1$, $3b = 1 + \frac{1}{4} = \frac{5}{4}$ and $b = \frac{1}{3} \cdot \frac{5}{4} = \frac{5}{12}$.

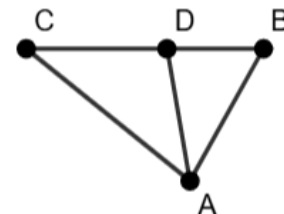
Finally, rearrange $a + b + c = 1$ as $c = 1 - a - b$ and substitute $a = -\frac{1}{60}$ and $b = \frac{5}{12}$:

$$\begin{aligned} c &= 1 - \left(-\frac{1}{60} \right) - \frac{5}{12} = \frac{60}{60} + \frac{1}{60} - \frac{25}{60} \\ &= \frac{60 + 1 - 25}{60} = \frac{36}{60} = \frac{6}{10} = \frac{3}{5} \end{aligned}$$

Collect the results: $(a, b, c) = \left(-\frac{1}{60}, \frac{5}{12}, \frac{3}{5} \right)$.

Team Round

1. In $\triangle ABC$, shown in the figure to the right, $\overline{AD}$ bisects $\angle CAB$ and D lies on $\overline{BC}$ between C and B . If $AC = 36$, $AB = 27$, and $BC = 42$, find BD .



Solution: Recall the Angle Bisector Theorem (also known as the Triangle Angle Bisector Theorem), which states that a ray that bisects an angle of a triangle will divide the opposite side in proportion to the lengths of the corresponding adjacent sides of the triangle. Apply the theorem to the given information: $\overline{AD}$ divides $\overline{BC}$ into $\overline{BD}$ and $\overline{CD}$ such that:

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

Let $BD = x$ (the value to be found). Note that $BC = BD + CD$, and that $CD = BC - BD = 42 - x$. Rewrite the proportion with BD isolated on the left and substitute known values:

$$BD = x = CD \cdot \frac{AB}{AC} = (42 - x) \frac{27}{36} = (42 - x) \frac{3}{4}.$$

Multiply both sides by $\frac{4}{3}$: $\frac{4}{3}x = 42 - x$. Add x to both sides: $x + \frac{4}{3}x = \frac{7}{3}x = 42$. Finally, $x = BD = \frac{3}{7} \cdot 42 = 3 \cdot \frac{42}{7} = 3 \cdot 6 = \boxed{18}$.

2. Penelope has dimes and quarters in her pocket worth a total of \$1.15. What is the difference between the largest number and the smallest number of dimes that she might have in her pocket?

Solution: Let d equal the number of dimes and let q equal the number of quarters in Penelope's pocket. Then $10d + 25q = 115$ because a dime is worth 10 cents, a quarter is worth 25 cents, and there are 115 cents in \$1.15. Divide the equation by 5 and:

$$2d + 5q = 23$$

Note that $2d$ is even and that $5q$ must be odd because only an odd plus an even will equal an odd (23 in this case). Therefore q must be odd; if it were even, then $5q$ would be even, which it is not.

The smallest value of q is 1 because 1 is the smallest odd number. The largest value of q is 3; if q were 5 or larger then $d = \frac{23-5q}{2}$ would be negative.

If $q = 1$, $2d = 23 - 5 = 18$ and $d = 9$. If $q = 3$, $2d = 23 - 5 \cdot 3 = 23 - 15 = 8$ and $d = 4$. Therefore the difference between the largest and smallest possible numbers of dimes is $9 - 4 = \boxed{5}$.

3. Let $f(x)$ and $g(x)$ be functions with domain $D = \{1, 2, 3, 4, 5\}$. They are defined by the following table:

x	1	2	3	4	5
$f(x)$	4	5	2	3	1
$g(x)$	3	1	4	5	2

Find $f(g^{-1}(f^{-1}(2)))$.

Solution: Evaluate the expression from the inside out. First evaluate $f^{-1}(2)$ by finding 2 in the $f(x)$ row and noting that it is the $x = 3$ column. Therefore $f(3) = 2$ and $f^{-1}(2) = 3$. Then $g^{-1}(f^{-1}(2)) = g^{-1}(3)$. Note that the 3 in the $g(x)$ row is in the $x = 1$ column, so that $g(1) = 3$ and $g^{-1}(3) = 1$. Finally, $f(g^{-1}(f^{-1}(2))) = f(1) = \boxed{4}$ because 4 is the number in the $f(x)$ row and the $x = 1$ column.

4. Point S has coordinates $(x, y) = (-2, 4)$. When S is rotated 90° counterclockwise around the point $(1, 1)$ its coordinates are (x_{90}, y_{90}) . Find (x_{90}, y_{90}) .

Solution: Call the point of rotation P . Recall how to perform a rotation about the origin. One way to find the rotation of S about P is to translate the rotation operation to the origin where the operation is more familiar. More specifically, follow the following process:

- translate P to the origin $O(0, 0)$ and translate S by the same vector to S'
- perform the rotation on S' about O , resulting in S''
- translate S'' by the translation (the inverse of the first translation) that will return O to P

The result of the translation, rotation, and inverse translation is the same as directly performing the rotation of S about P .

Follow this plan: translate $P(1, 1)$ by the vector $V(-1, -1)$ to $O(0, 0)$ and translate S to $S + V = (-2, 4) + (-1, -1) = (-2 - 1, 4 - 1) = (-3, 3) = S'$. Note that S' lies in Quadrant II and $\overline{OS'}$ makes an angle of 45° with the x -axis. Therefore a rotation of 90° will place S'' in Quadrant III where $\overline{OS''}$ makes an angle of $90 - 45 = 45^\circ$ with the x -axis. Recall that both x and y are negative in Quadrant III, and that S'' lies on the line $x = y$ due to the 45° angle. Also, rotation does not change the distance to the point of rotation (O). Therefore S'' has coordinates $(-3, -3)$.

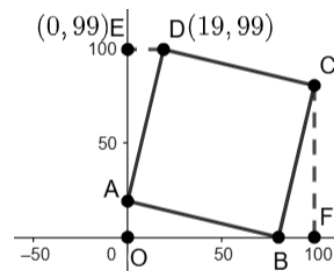
Finally, reverse the translation, and $S''' = S'' + -(-1, -1) = (-3, -3) + (1, 1) = \boxed{(-2, -2)}$

5. Square $ABCD$ is located in the first quadrant of the x - y plane. A lies on the y -axis, B lies on the x -axis, and the coordinates of D are $(19, 99)$. Find (x_C, y_C) , the coordinates of C .

Solution: First sketch $ABCD$ on the x, y plane as shown at right. Label the origin $O(0, 0)$. Construct a line segment $\overline{DE}$ parallel to the x -axis and perpendicular to the y -axis that intersects the y -axis at $E(0, 99)$, as shown in the diagram at right. Note that $\angle DEA \cong \angle AOB$ because they are both right angles. Note also that $\overline{DA} \cong \overline{AB}$ because they are sides of square $ABCD$. Next, note that $\angle OAB$ is complementary to both $\angle ABO$ (because they are the acute angles of right triangle $\triangle AOB$) and $\angle DAE$ (because $m\angle DAE + m\angle DAB + m\angle OAB = 180^\circ$ and $m\angle DAB = 90^\circ$). Therefore $\angle EDA \cong \angle OAB$. Conclude that $\triangle DEA \cong \triangle AOB$ by AAS congruence.

Note that $\overline{DE} \cong \overline{AO}$ and $\overline{EA} \cong \overline{OB}$ because they are two pairs of corresponding sides of congruent triangles. Also note that $DE = 19 = AO$ because $\overline{DE}$ extends from $(19, 99)$ to $(0, 99)$. Then A has coordinates $(0, 19)$ and $EA = 99 - 19 = 80 = OB$ because $\overline{EA}$ extends from $(0, 99)$ to $(0, 19)$. Therefore B has coordinates $(80, 0)$.

Construct a line segment $\overline{CF}$ parallel to the y -axis and perpendicular to the x -axis that intersects the x -axis at $F(x_C, 0)$. Observe that $\triangle BFC \cong \triangle AOB$ by an argument similar to the one above used to prove $\triangle DEA \cong \triangle AOB$. Then $BF = AO = 19$ and $FC = OB = 80$. Therefore $y_C = OB = 80$ and $x_C = OB + BF = 80 + 19 = 99$. The coordinates of C are $(x_C, y_C) = \boxed{(99, 80)}$.



6. At 8am Andrew started driving his Audi due North from his home in Beantown on Route 314. At 12 noon Cindy started driving her Cadillac from Andrew's home, following Andrew on Route 314. At 5pm Cindy had passed Andrew and was 50 miles ahead of him. If Andrew averaged 25 miles per hour (mph) for his drive, what was Cindy's average speed, measured in mph?

Solution: Andrew drove for four hours from 8am to noon and then for five hours from noon to 5pm for a total of $4 + 5 = 9$ hours, traveling at 25mph on average. He therefore drove $25 \cdot 9 = 225$ miles. Cindy drove for five hours from noon to 5pm for a distance of $225 + 50 = 275$ miles. Cindy's average speed was therefore $\frac{275}{5} = \boxed{55}$ mph.

7. If $f(x) = 2x + 3$, then $f^{-1}(x) = ax + b$, where $f^{-1}(x)$ is the inverse of $f(x)$. Find the ordered pair (a, b) where a and b are expressed as fractions.

Solution: Recall that if $f^{-1}(x)$ is the inverse of $f(x)$, then $f(f^{-1}(x)) = x$ for all x in the domain of $f^{-1}(x)$. Substitute $f^{-1}(x) = ax + b$ into $f(f^{-1}(x)) = x$ and solve for a and b :

$$f(f^{-1}(x)) = f(ax + b) = 2(ax + b) + 3 = 2ax + 2b + 3 = x$$

Subtract x from both sides of the equation and $(2a - 1)x + 2b + 3 = 0$. For this equality to hold for all x , both the expression multiplying x and the constant term must equal zero. Therefore

$$2a - 1 = 0, 2a = 1, \text{ and } a = \frac{1}{2}. \text{ Also } 2b + 3 = 0, \text{ and } b = -\frac{3}{2}. \text{ Thus, } (a, b) = \boxed{\left(\frac{1}{2}, -\frac{3}{2}\right)}.$$

8. How many ways can three boys and two girls sit in a row of five chairs if the two girls always sit next to each other?

Solution: Begin by treating the two girls as if they were a single person. Recall that there are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways of ordering four items, where in this problem the four items are the three boys and the pair of girls. For each of these orderings there are two ways to order the two girls. In total there are $2 \cdot 24 = \boxed{48}$ ways to seat the three boys and two girls with the given constraint.

9. What is the radius of the circle that passes through the points $(0,0)$, $(1,1)$, and $(11,1)$? Give your answer in simplest radical form.

Solution: Name the three points $A(0,0)$, $B(1,1)$, and $C(11,1)$. Recall that the perpendicular bisectors of the three sides of a triangle intersect in a single point, the circumcenter, which is the center of the circle that circumscribes the triangle. Let l_{AB} be the perpendicular bisector of $\overline{AB}$ and let l_{BC} be the perpendicular bisector of $\overline{BC}$. Proceed by finding the equations of l_{AB} and l_{BC} . Calculate the midpoint of $\overline{AB}$: $(\frac{0+1}{2}, \frac{0+1}{2}) = (\frac{1}{2}, \frac{1}{2})$. Calculate the slope of AB : $m_{AB} = \frac{1-0}{1-0} = 1$. Recall that the product of the slopes of two perpendicular lines is equal to -1 , so the slope of the perpendicular bisector of $\overline{AB}$ is equal to -1 . Apply the point-slope formula to find the equation of l_{AB} , where $(\frac{1}{2}, \frac{1}{2})$ lies on l_{AB} :

$$m = -1 = \frac{y - \frac{1}{2}}{x - \frac{1}{2}}$$

Multiply both sides of this equation by $x - \frac{1}{2}$: $\frac{1}{2} - x = y - \frac{1}{2}$, or $x + y = \frac{1}{2} + \frac{1}{2} = 1$.

Apply the midpoint formula to calculate the midpoint of $\overline{BC}$: $(\frac{1+11}{2}, \frac{1+1}{2}) = (6, 1)$. Note that the slope of $\overline{BC}$ is zero and that it is parallel to the x -axis. Therefore l_{BC} is perpendicular to the y -axis, and its equation is $x = 6$. The x -coordinate of the center of the circle is 6 since it must lie on l_{BC} .

Substitute $x = 6$ into the l_{AB} equation: $6 + y = 1$. Then $y = 1 - 6 = -5$ and the center of the circle is $(6, -5)$. Finally, calculate the radius by finding the distance from the center to point $A(0,0)$, applying the distance formula: $r = \sqrt{(6-0)^2 + (-5-0)^2} = \sqrt{6^2 + (-5)^2} = \sqrt{36 + 25} = \boxed{\sqrt{61}}$.